Finite-valued Streaming String Transducers

Emmanuel Filiot, Ismaël Jecker, Christof Löding,
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Finite-valued regular relations are binary relations that enjoy good algorithmic properties

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Theorem: We can decide in polynomial space whether a given **SST** defines a **finite-valued regular relation**

Theorem: Every **finite-valued regular relation** can be decomposed into a finite union of **regular functions**

TRANSDUCERS

Transducers are abstract machines that recognise **relations**

$$\mathsf{R} \subseteq \Sigma^* \times \Gamma^*$$

Rational relations	recognised by finite state transducers
Regular relations	recognised by streaming string transducers

Transducers

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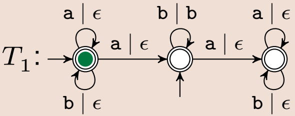
Output:



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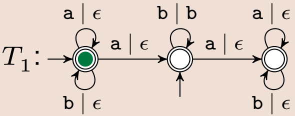

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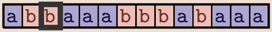
Input: 

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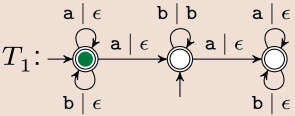

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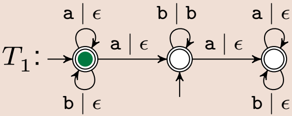

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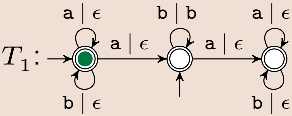

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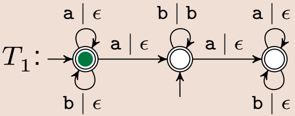

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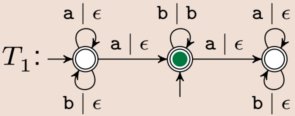

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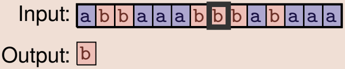

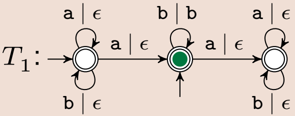

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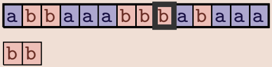

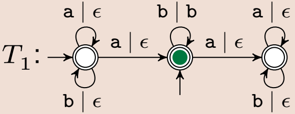

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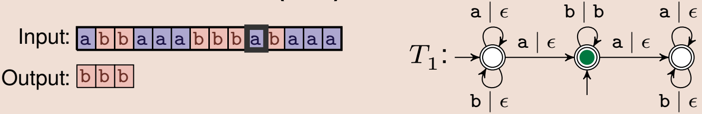

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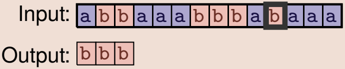

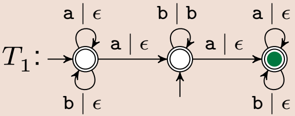

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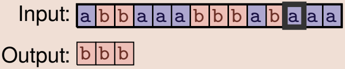

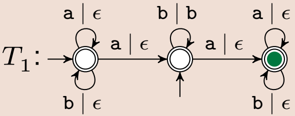

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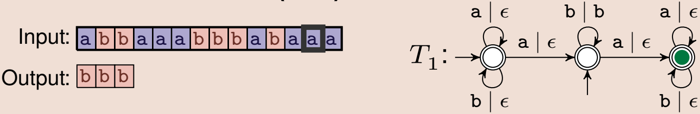

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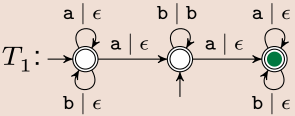
Finite state transducer (FST)

Input: a b b a a a b b b a b a a a

Output: b b b



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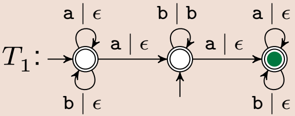

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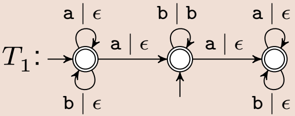
Output: b b b



T_1 produces one of the b-blocks of its input

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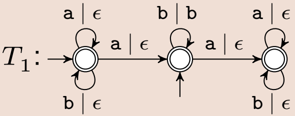
Input: a b b a a a b b b a b a a a

Output: b b b   b   b b



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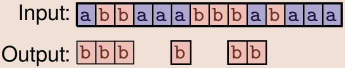

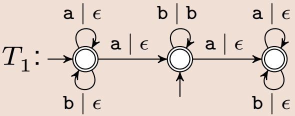

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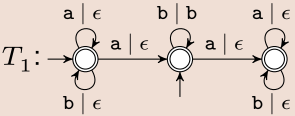
Finite state transducer (FST)

Input: abbaaabbbabaaa

Output: bbb b bb



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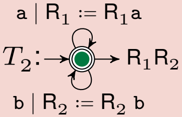
Streaming string transducer (SST)

Input: a b b a a a b b b a b a a a

R_1:

R_2:



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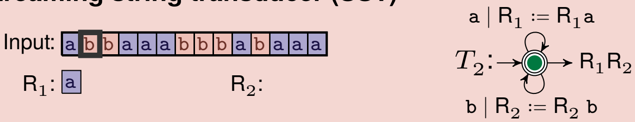

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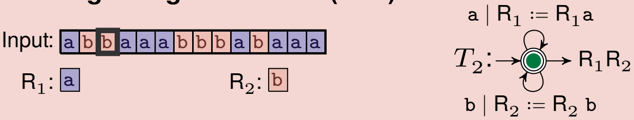

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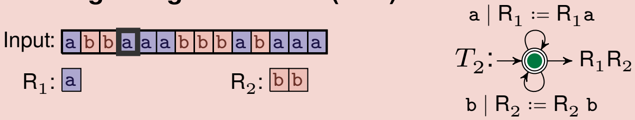

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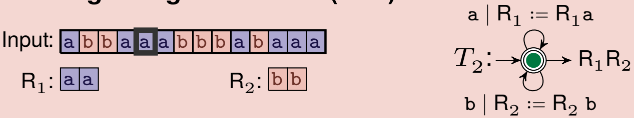

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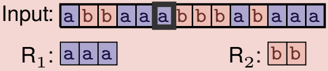

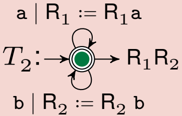

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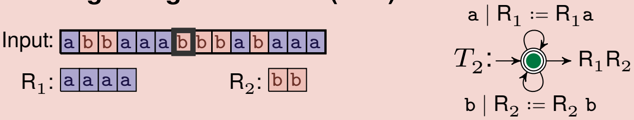

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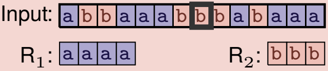

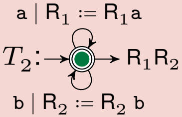

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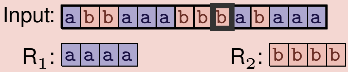

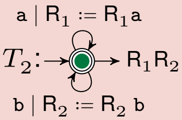

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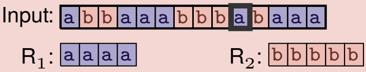

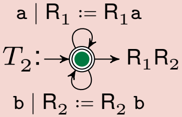

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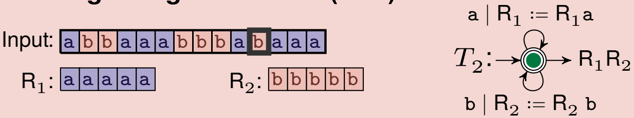

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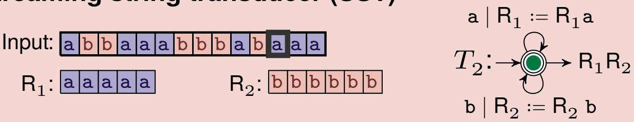

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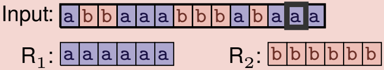

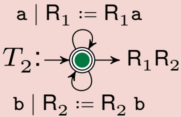

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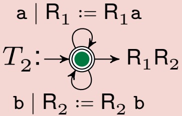
Streaming string transducer (SST)

Input: a b b a a a b b b a b a a a

R_1: a a a a a a a

R_2: b b b b b b



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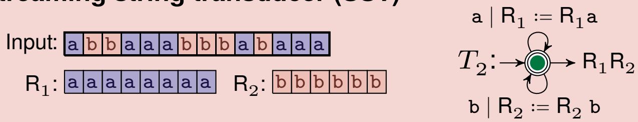

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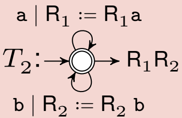
Output: a a a a a a a a b b b b b b



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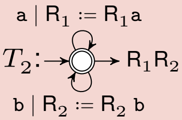

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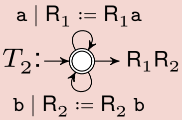
Output: a a a a a a a a b b b b b b

T_2 sorts its input



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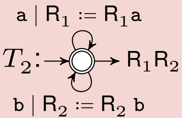

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Equivalence of **rational relations** is **undecidable**

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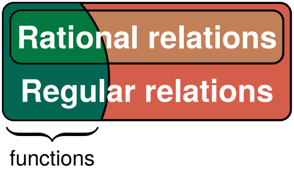
Equivalence of **rational relations** is **undecidable**

Equivalence of **regular functions** is **decidable**

↳ each input is mapped to at most 1 output



recognised by finite state transducers

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Transducers

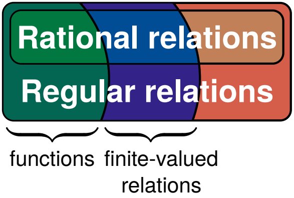
Equivalence of **rational relations** is **undecidable**

Equivalence of **regular functions** is **decidable**

↳ each input is mapped to at most 1 output

Equivalence of **finite-valued regular relations** is **decidable**

↳ $\exists \mathbf{k} \in \mathbb{N}$ s.t. each input is mapped to at most **k** outputs



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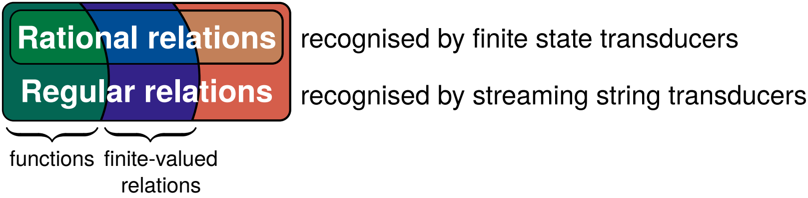
Transducers

Theorem: We can decide in polynomial space whether a given **SST** defines a **finite-valued regular relation**

Theorem: Every **finite-valued regular relation** can be decomposed into a finite union of **regular functions**



TRANSDUCERS

Theorem: We can decide in polynomial space whether a given **SST** defines a **finite-valued regular relation**

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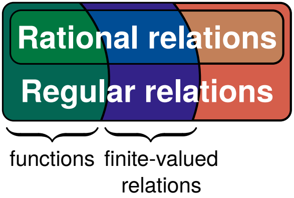
⇒ The equivalence problem for finite-valued SST is in ELEMENTARY

⇒ Finite-valued **2-way FST** are as expressive as finite-valued SST



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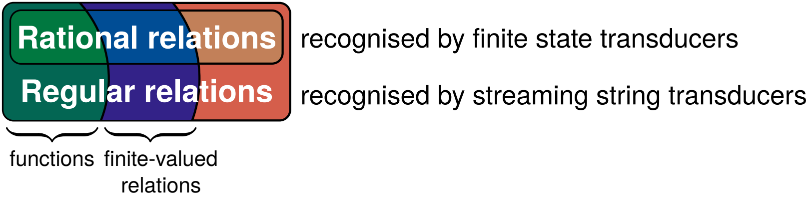

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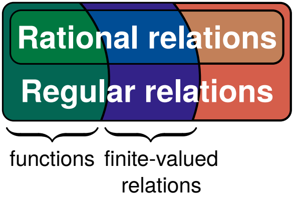
Raised as open problems by [2011. Alur, Deshmukh]

Known to hold for FST [1989. Weber], [1993. Weber]

...and for SST with a single register [2017. Gallot et al.]



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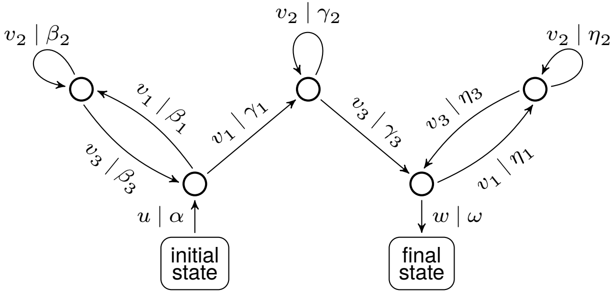
Deciding Finite Valuedness

Theorem: We can decide in polynomial space whether a given SST defines a **finite-valued regular relation**

Forbidden pattern: (inspired by [2008. De Souza])



- The relation recognised by the pattern is not 1-valued
- The substitutions produced on the loops have idempotent structure

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Proof: (inspired by [2008. Sakarovitch, de Souza] and relying on [2023. FJLW])

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$\mathcal{T}$ that associates at most **k** outputs to each input

$\mathcal{T}_1$, ..., $\mathcal{T}_k$ that all associate at most **1** output to each input

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$\mathcal{T}$' that has at most **k** runs on each input

$\mathcal{T}'_i$ copies the ith lexicographically smallest run of $\mathcal{T}'$

$\mathcal{T}'_1$, ..., $\mathcal{T}'_k$ that all have at most **1** run on each input

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$\Downarrow$

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⇓ $\mathcal{T}'$ only keeps the runs of $\mathcal{T}$ that are **far** from each other

$\mathcal{T}$' that has at most **k** runs on each input

⇓

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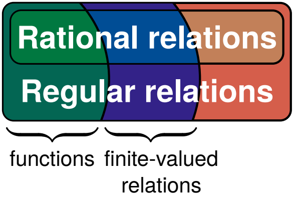

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