Finite-valued Streaming String Transducers

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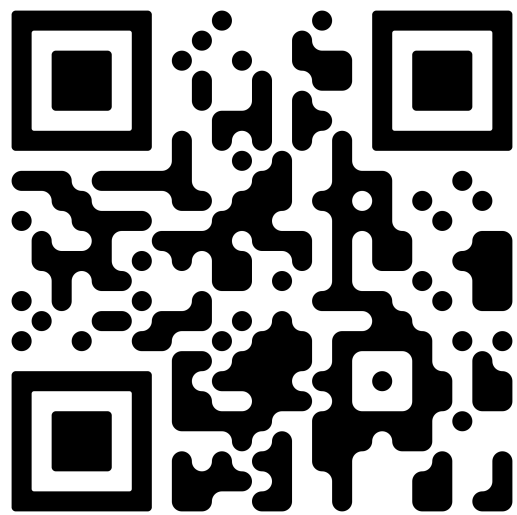
Anca Muscholl, Gabriele Puppis, Sarah Winter

ismaeljecker.github.io

Finite-valued regular relations form a class of binary relations that enjoys good algorithmic properties



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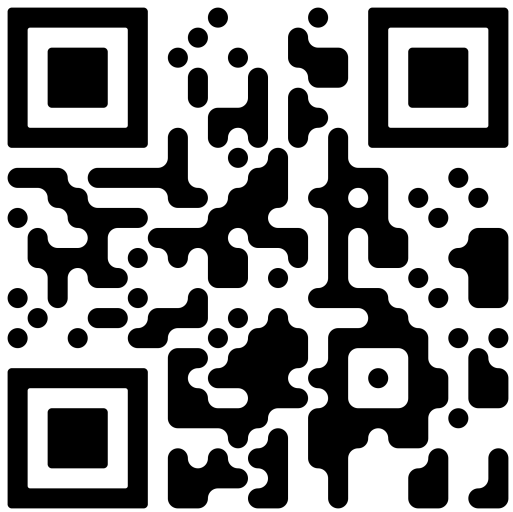
Theorem: We can decide in polynomial space whether a given **SST** defines a **finite-valued regular relation**

Theorem: Every **finite-valued regular relation** can be decomposed into a finite union of **regular functions**



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Transducers

Transducers are abstract machines that recognise **relations**

$$R \subseteq \Sigma^* \times \Gamma^*$$

Rational relations recognised by finite state transducers

Regular relations recognised by streaming string transducers

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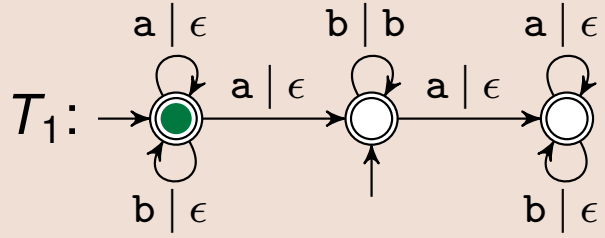

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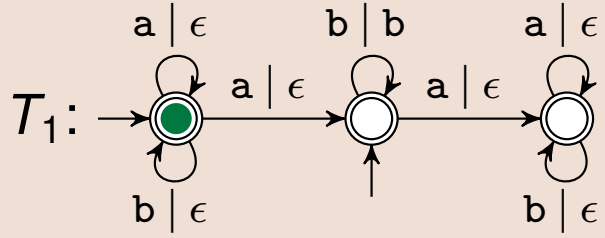

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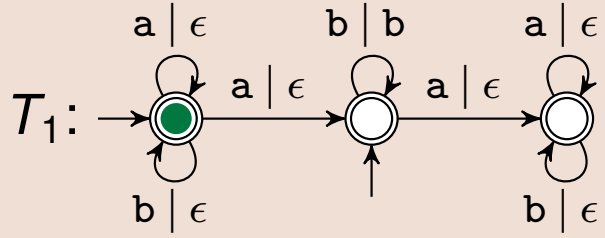

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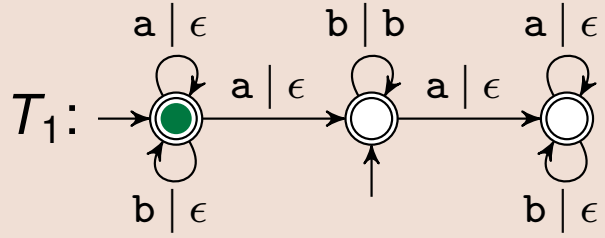

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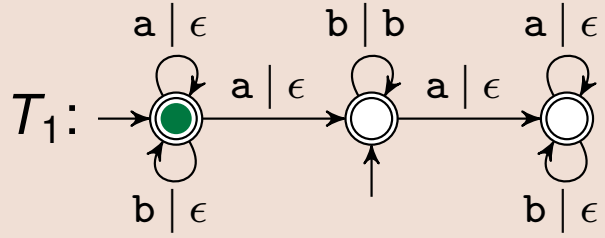

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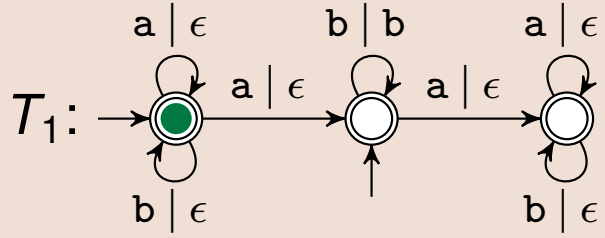

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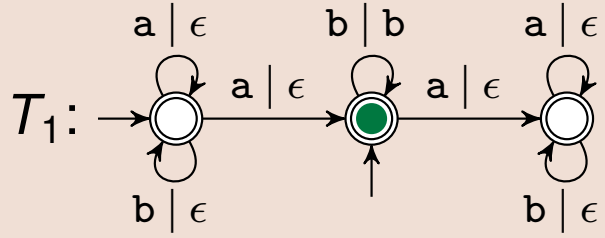

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Input: a b b a a a b b b a b a a a

Output: b

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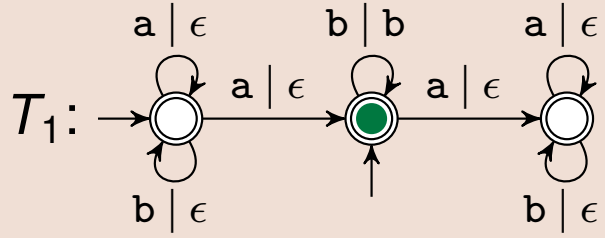

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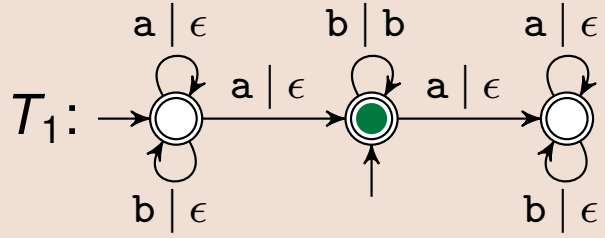

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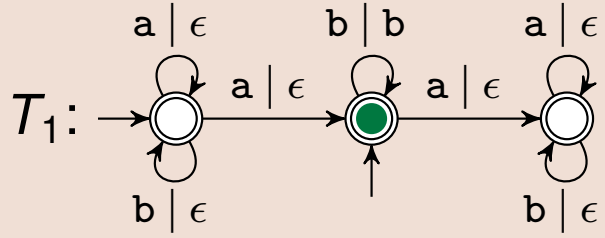

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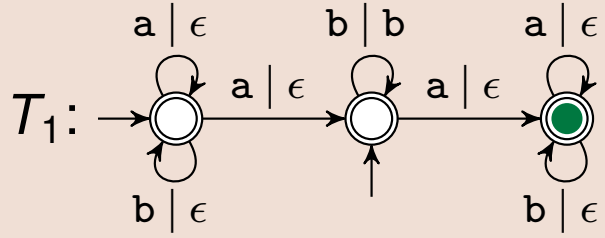

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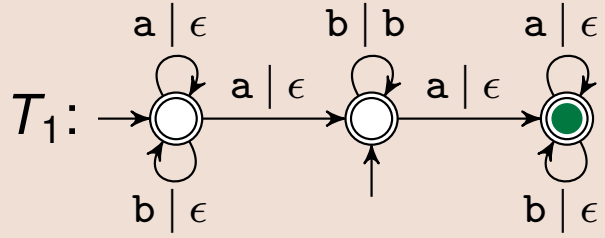

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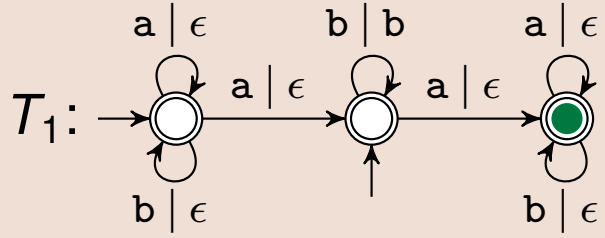

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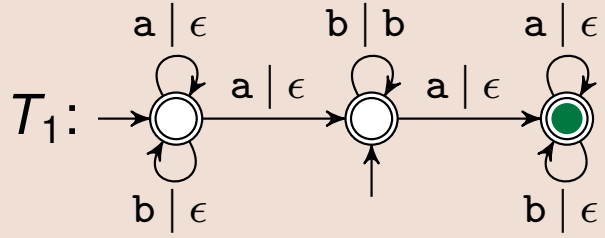

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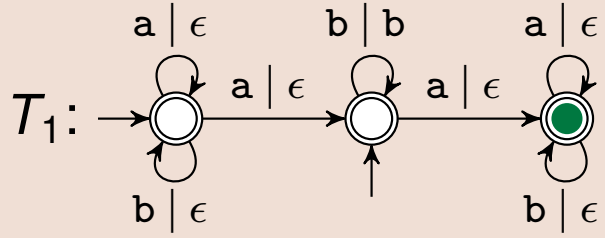

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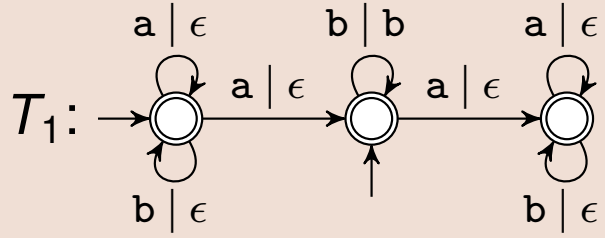

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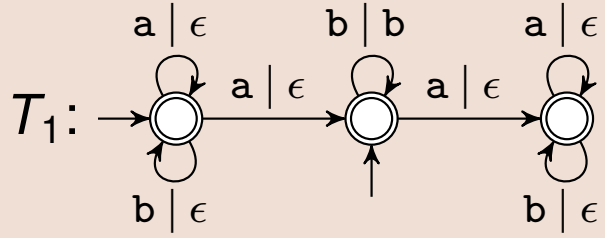

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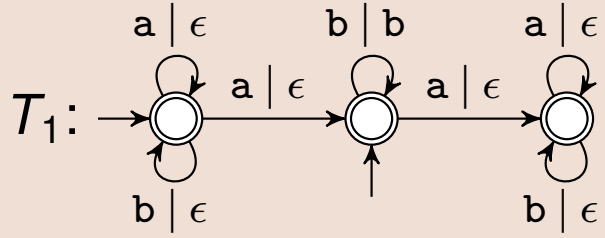

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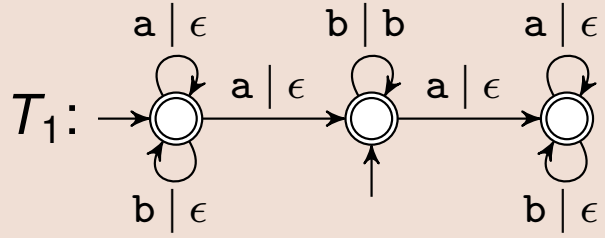

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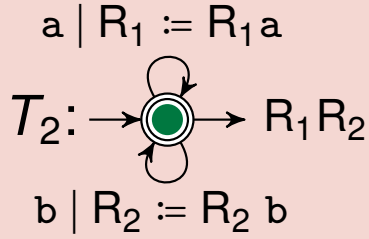
Streaming string transducer (SST)

Input: a b b a a a b b b a b a a a

R_1:

R_2:

T_2 sorts its input



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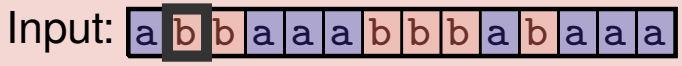

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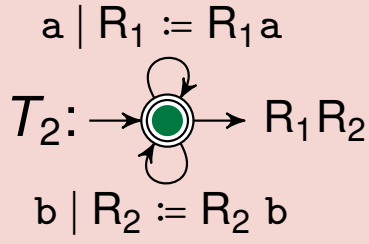

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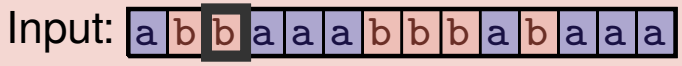

R_1: a R_2: b

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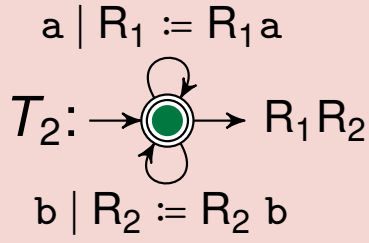

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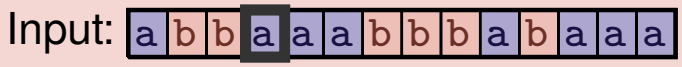

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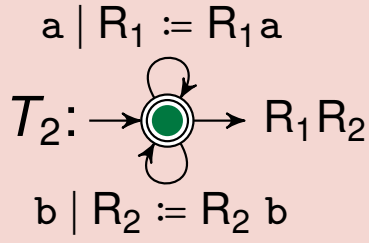

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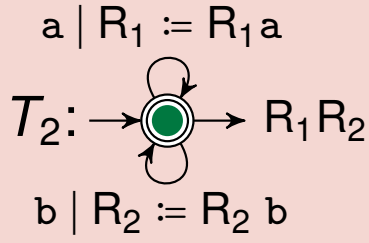

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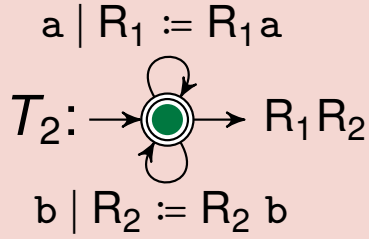

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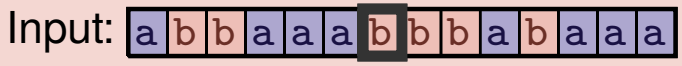

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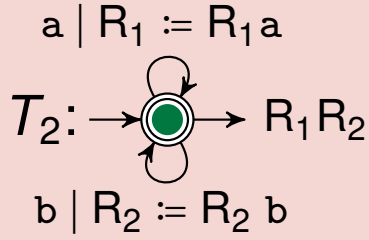

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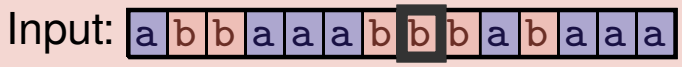

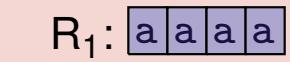

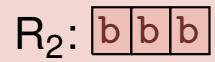

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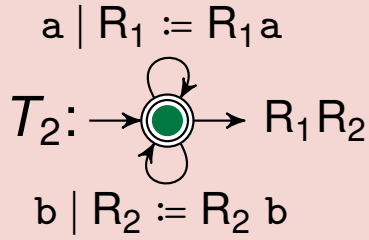

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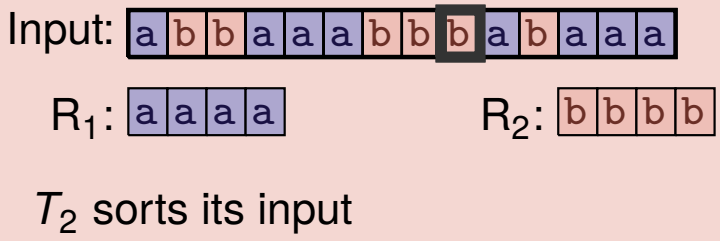

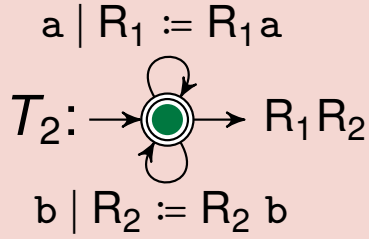

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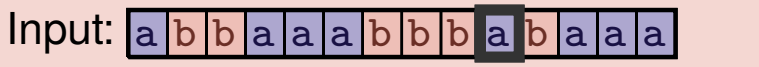

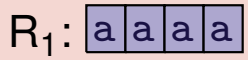

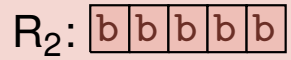

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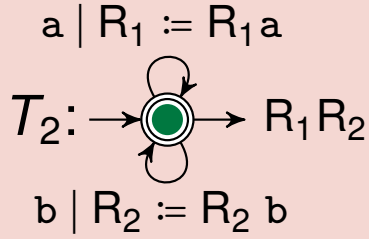

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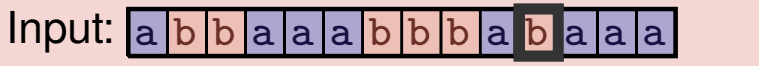
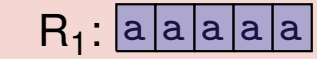
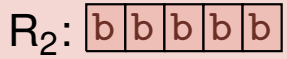
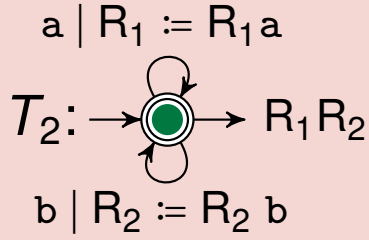






T_2 sorts its input



Rational relations recognised by finite state transducers

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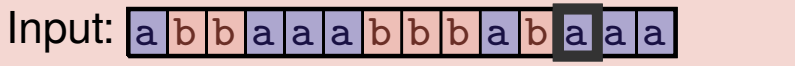
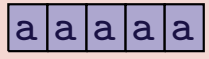
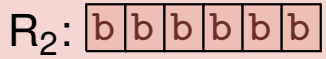
Transducers

Transducers are abstract machines that recognise **relations**

Streaming string transducer (SST)







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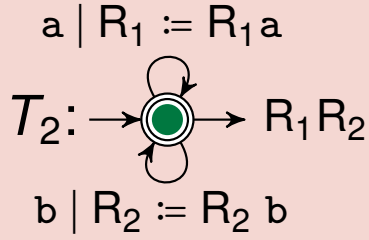

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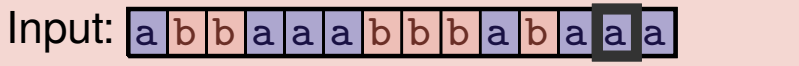

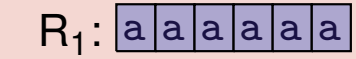

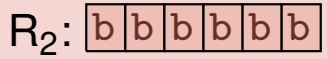

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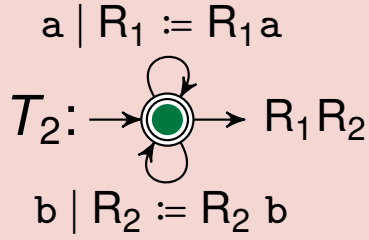

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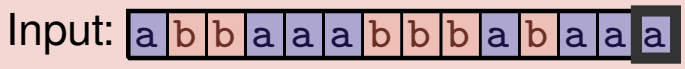

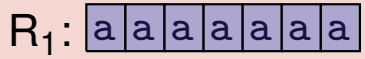

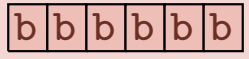

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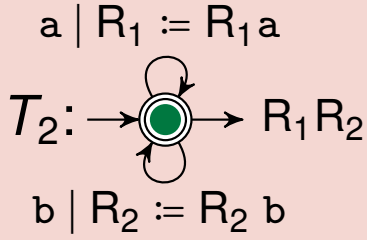

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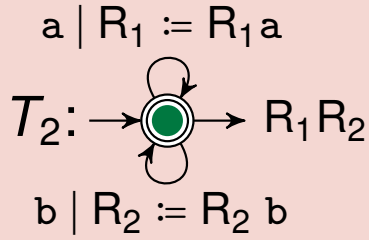
Streaming string transducer (SST)

Input: a b b a a a b b b a b a a a

R_1: a a a a a a a a R_2: b b b b b b

T_2 sorts its input



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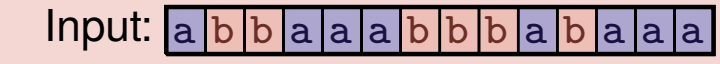

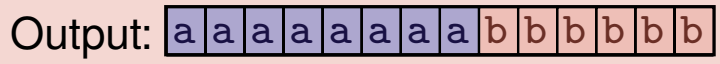

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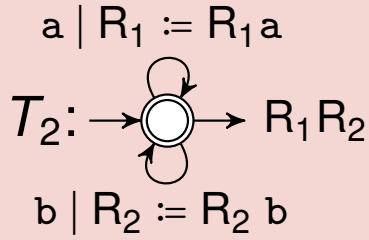

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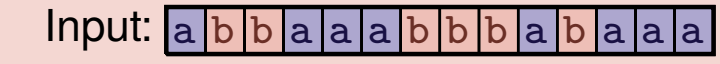

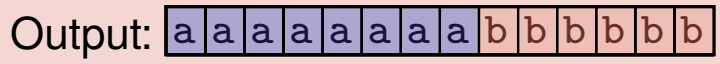

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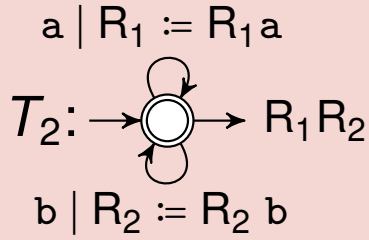

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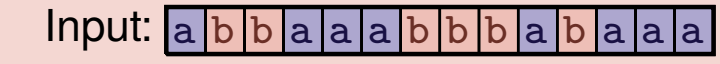

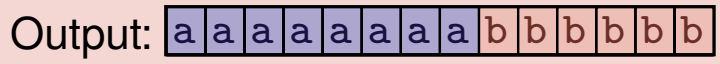

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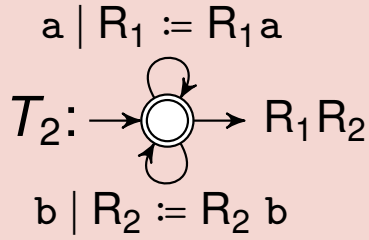

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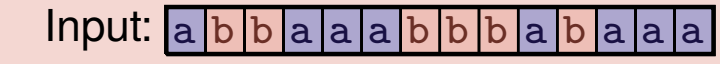

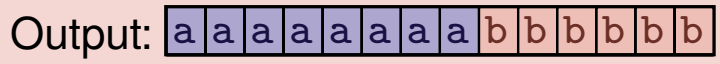

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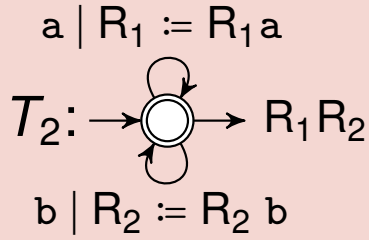

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Transducers

Equivalence of **rational relations** is **undecidable**

Rational relations recognised by finite state transducers

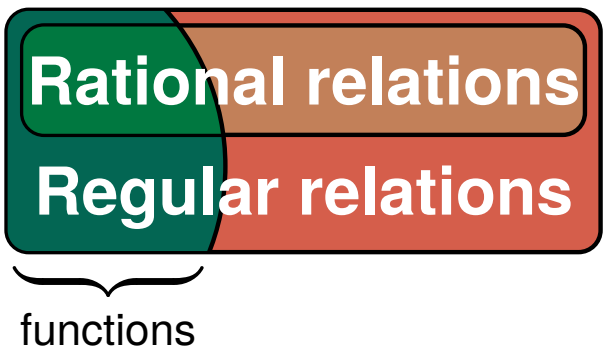
Regular relations recognised by streaming string transducers

Transducers

Equivalence of **rational relations** is **undecidable**

Equivalence of **regular functions** is **decidable**

↳ each input is mapped to at most 1 output



recognised by finite state transducers

recognised by streaming string transducers

Transducers

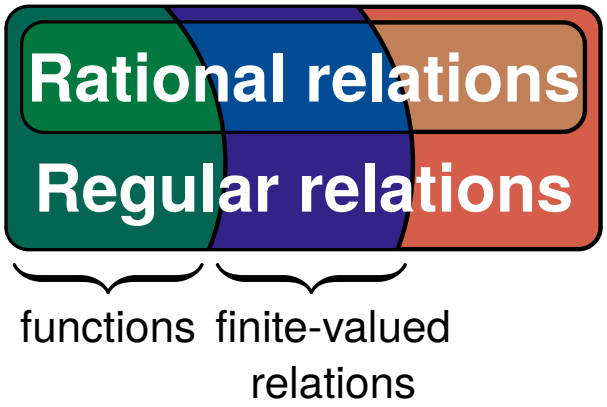
Equivalence of **rational relations** is **undecidable**

Equivalence of **regular functions** is **decidable**

↳ each input is mapped to at most 1 output

Equivalence of **finite-valued regular relations** is **decidable**

↳ $\exists \mathbf{k} \in \mathbb{N}$ s.t. each input is mapped to at most **k** outputs



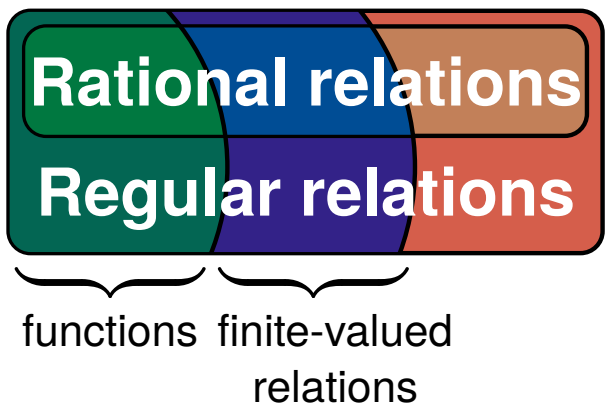
recognised by finite state transducers

recognised by streaming string transducers

Transducers

Theorem: We can decide in polynomial space whether a given **SST** defines a **finite-valued regular relation**

Theorem: Every **finite-valued regular relation** can be decomposed into a finite union of **regular functions**



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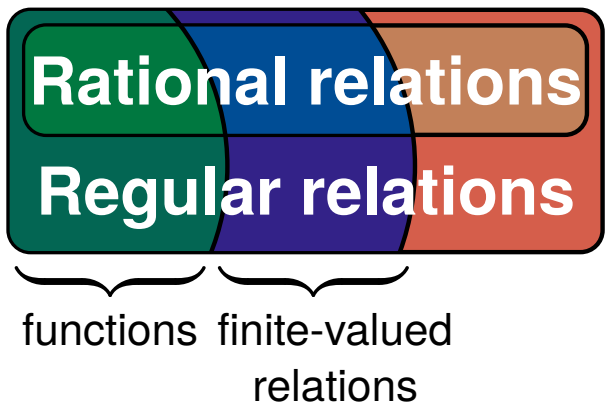
Theorem: We can decide in polynomial space whether a given **SST** defines a **finite-valued regular relation**

Theorem: Every **finite-valued regular relation** can be decomposed into a finite union of **regular functions**

$\Rightarrow$ The equivalence problem for finite-valued SST is in Elementary

$\Rightarrow$ Finite-valued **2-way FST** are as expressive as finite-valued SST



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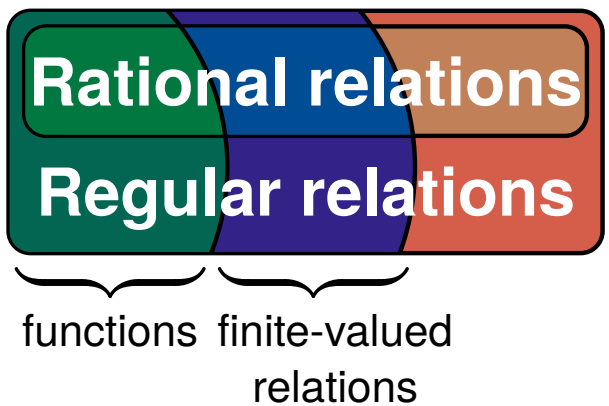

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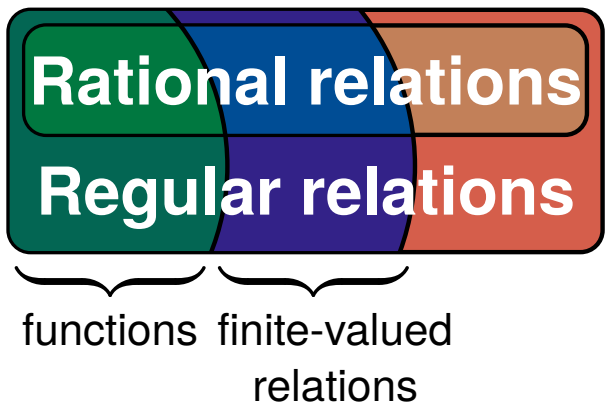
Raised as open problems by [2011. Alur, Deshmukh]

Known to hold for FST [1989. Weber], [1993. Weber];

...and for SST with a single register [2017. Gallot et al.]



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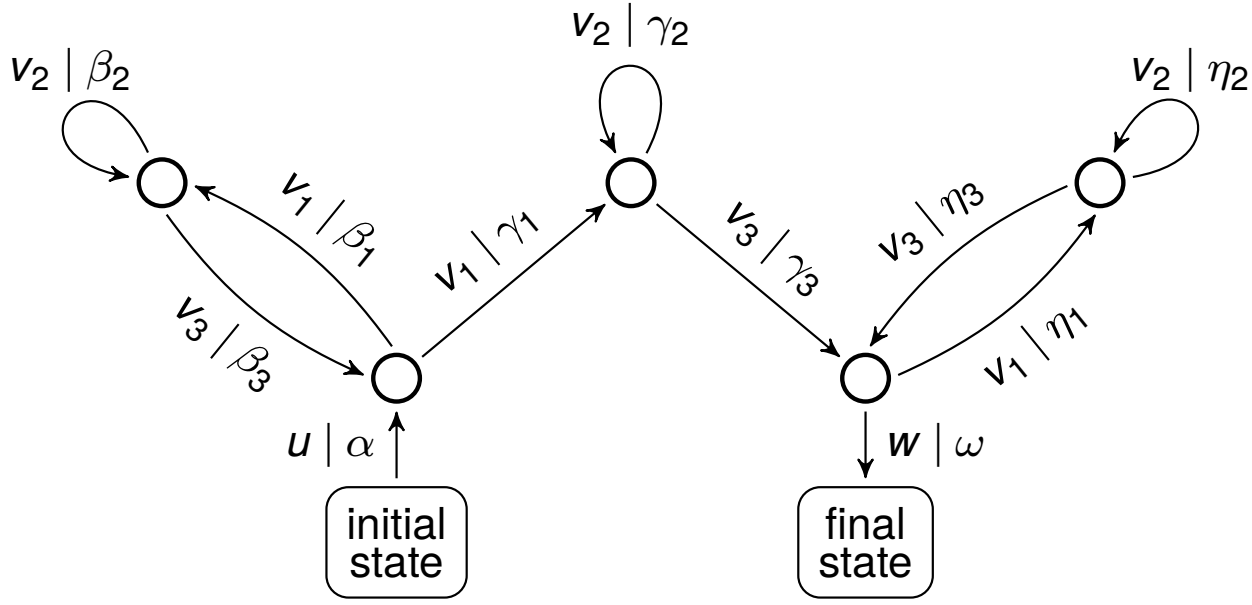
Deciding Finite Valuedness

Theorem: We can decide in polynomial space whether a given SST defines a **finite-valued regular relation**

Forbidden pattern: (inspired by [2008. De Souza])



- The relation recognised by the pattern is not 1-valued
- The substitutions produced on the loops have idempotent structure

The Decomposition Theorem

Theorem: Every **finite-valued regular relation** can be decomposed into a finite union of **regular functions**

Proof: (inspired by [2008. Sakarovitch, de Souza] and relying on [2023. FJLW])

$\mathcal{T}$ **k**-valued (at most **k** outputs per input)

$\mathcal{T}'$ **k**-ambiguous (at most **k** runs per input)

$\mathcal{T}'_1, \ldots, \mathcal{T}'_k$ unambiguous (at most **1** run per input)

$\mathcal{T}_1, \ldots, \mathcal{T}_k$ functional (at most **1** output per input)

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$\mathcal{T}$ **k**-valued (at most **k** outputs per input)

⇓ $\mathcal{T}'$ only keeps the runs of $\mathcal{T}$ that are **far** from each other

$\mathcal{T}'$ **k**-ambiguous (at most **k** runs per input)

⇓

$\mathcal{T}'_1, \ldots, \mathcal{T}'_k$ unambiguous (at most **1** run per input)

⇓

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$\mathcal{T}$ **k**-valued (at most **k** outputs per input)

$\mathcal{T}'$ **k**-ambiguous (at most **k** runs per input)

$\mathcal{T}'_i$ copies the ith lexicographically smallest run of $\mathcal{T}'$

$\mathcal{T}'_1, \ldots, \mathcal{T}'_k$ unambiguous (at most **1** run per input)

$\mathcal{T}_1, \ldots, \mathcal{T}_k$ functional (at most **1** output per input)

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$\mathcal{T}'$ **k**-ambiguous (at most **k** runs per input)

$\mathcal{T}'_1, \ldots, \mathcal{T}'_k$ unambiguous (at most **1** run per input)

$\mathcal{T}_i = \mathcal{T}'_i$

$\mathcal{T}_1, \ldots, \mathcal{T}_k$ functional (at most **1** output per input)

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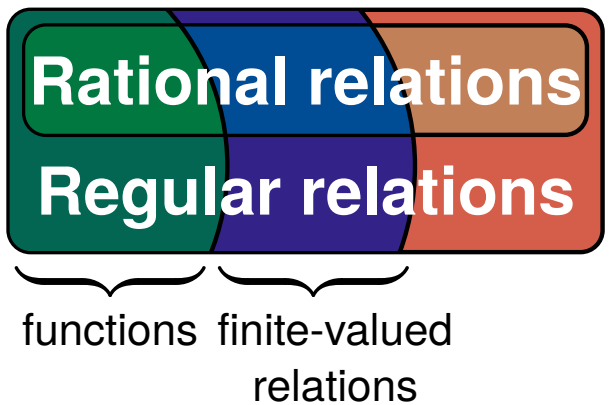

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Besançon is hiring

The University of Franche-Comté is recruiting for

3 Maître de conférences positions

set to begin in the academic year 2025-2026

Contact me for more information:

ismael.jecker@gmail.com